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IGNITING A PLANE WEDGE BY A GLOWING SURFACE*

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A method is proposed for the asymptotic analysis of qualitative characteristics of the process of igniting rough surface regions occupied by a reacting medium. The problem of igniting a plane wedge whose faces are maintained at constant temperature with the roughness of reacting surfaces taken into account, is taken as the model. The chemical reaction activation energy is assumed fairly high, as is the case with the majority of exothermic reacting media.

In classical problems of the thermal theory of ignition by a glowing surface the multi-dimensionality of combustible substance specimens is not taken into account /1,2/. It is usual to consider the ignition of a half-space occupied by the reacting medium whose surface temperature is maintained constant. However in real systems the ignited surface of the reacting medium can contain various irregularities (protrusions, cracks, etc.) that play the part of leaders or outsiders in the process. The conditions for igniting surfaces with protrusions are more favorable than for those that have plane /smooth/ surfaces, since at the protrusion side faces the heat flux is directed into the latter. The effect of recesses in the reacting medium surface is opposite.

1. Basic equations. Consider a two-dimensional region of the form of a wedge with an apex angle α, filled by a substance susceptible to chemical exothermic transformations. Constant temperture T_+ is maintained at the wedge surface. Let us determine the ignition characteristics and temperature distribution in the wedge-shaped region occupied by the reacting substance. Neglecting the reagent burnout, we define the problem of temperature distribution at the corner as follows:

$$\frac{\partial \Theta}{\partial t} = \Delta\Theta + \varepsilon^{-1} \exp\left[\frac{\varepsilon^{-1}(1+\sigma)(\Theta-1)}{\Theta+\sigma}\right] \tag{1.1}$$

$$\Theta(0, \eta, \varphi) = \Theta(t, \infty, \varphi) = 0 \tag{1.2}$$

$$\Theta(t, \eta, 0) = \Theta(t, \eta, \alpha) = 1$$

$$\Theta = \frac{T - T_-}{T_+ - T_-}, \quad t = \frac{t'}{\Delta t}, \quad X = \frac{x'}{\Delta x}, \quad Y = \frac{y'}{\Delta x}, \quad \eta = \frac{\rho}{\Delta x}$$

$$(\Delta x)^2 = \frac{\lambda}{\rho c \Delta t}, \quad \Delta t = \frac{E(T_+ - T_-)}{T_+^2 k Q} \exp\left(\frac{E}{RT_+}\right)$$

$$\varepsilon = \frac{RT_+^2}{E(T_+ - T_-)}, \quad \sigma = \frac{T_-}{T_+ - T_-}$$

$$\Delta = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial}{\partial \eta}\right) + \frac{1}{\eta^2}\frac{\partial^2}{\partial \varphi^2}$$

where x' and y' are orthogonal Cartesian coordinates with the x' axis coincident with one of the wedge sides, α is the apex angle, ρ' and φ are polar coordinates, t' is the time, T the temperature, ρ the density, c the specific heat, λ the thermal conductivity, Q the heat of combustion, E the activation energy, k the preexponentila factor, and R is the gas constant.

Let us assume that $0 < \varepsilon \ll 1$ and $RT_+/E \ll 1$. We seek a solution of system (1.1), (1.2) of the form

$$\Theta(t, \eta, \varphi, \varepsilon) = \Theta_i(t, \eta, \varphi) + U(t, \eta, \varphi, \varepsilon) \tag{1.3}$$

where Θ_i is a solution of form /3/

$$\Theta_i(t, \eta, \varphi) = 1 - \frac{4}{\alpha}\sum_{n=0}^{\infty} \sin[(2n+1)\nu\varphi] \int_0^{\infty} \xi^{-1} \exp(-\xi^2) \times J_{(2n+1)\nu}(\xi\eta t^{-1/2})\, d\xi, \quad \nu = \frac{\pi}{\alpha}$$

where $J_\nu(Z)$ is the Bessel function, of the inert problem $\partial\Theta_i/\partial t = \Delta\Theta_i$ with initial and boundary conditions (1.2).

When $\alpha = \pi/2$ ($\nu = 2$), we have

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$$\Theta_i(t, X, Y) = 1 - \operatorname{erf}(X/2\sqrt{t})\operatorname{erf}(Y/2\sqrt{t}) \tag{1.4}$$
$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-s^2)\,ds$$

Function U is the solution of the problem

$$\frac{\partial U}{\partial t} = \Delta U + \varepsilon^{-1}\exp\left[\frac{(1+\sigma)(\Theta_i + U - 1)}{\varepsilon(\Theta_i + U + \sigma)}\right] \tag{1.5}$$
$$U(0, \eta, \varphi) = U(t, \infty, \varphi) = U(t, \eta, 0) = U(t, \eta, \alpha) = 0$$

The reaction process is most intensive when $0 < \alpha < \pi$, hence we investigate the domain of parameter variation $1 \leqslant \nu$.

When $\nu = 1$, the problem reduces to the investigation of igniting by a glowing plane surface which was carried out in /1,2/. The asymptotic analysis of this problem (*) shows that the form of solution of (1.1), (1.2) changes when $t > t_i$, where

$$t_i(\varepsilon) = (2\pi)^{-1} + o(1) \tag{1.6}$$

This time is taken as the ignition time. A similar situation arises also when $1 < \nu$ at a fairly large distance from the wedge tip.

It is most natural to take as the wedge ignition time the instant at which the temperature at any point of the wedge interior exceeds unity.

Acceleration of the process is possible in the neighborhood of the tip. As implied by the form of the last term in the right-hand side of Eq.(1.5), it has an appreciable effect only when $(\Theta_i - 1)\varepsilon^{-1} = O(1)$.

With fixed t and φ and $\eta \to 0$ we have

$$\Theta_i(t, \eta \to 0, \varphi) = 1 - \frac{4^{(1-\nu)}}{\sqrt{\pi}}\Gamma^{-1}\left(\frac{\nu+1}{2}\right)\left(\frac{\eta}{\sqrt{t}}\right)^{\nu}\sin(\nu\varphi) + o\left(\left(\frac{\eta}{\sqrt{t}}\right)^{\nu}\sin(\nu\varphi)\right) \tag{1.7}$$

where $\Gamma(t)$ is the gamma function.

We introduce in the tip neighborhood the new three-dimensional variable $r = \eta/\varepsilon$, and shall consider the process in times of the order of $\delta^{-1}(\varepsilon)$ introducing for this the new time variable

$$\tau = t\delta(\varepsilon), \quad \delta \gg 1 \tag{1.8}$$

We seek in this region the solution $U(t, \eta, \varphi)$ of the form

$$U = \varepsilon U_1(\tau, r, \varphi) + \varepsilon^2 U_2(\tau, r, \varphi) + o(\varepsilon^2) \tag{1.9}$$

From (1.5) and (1.7)—(1.9) follows that

$$\varepsilon\delta\frac{\partial U_1}{\partial \tau} = \varepsilon^{-1}\Delta_r U_1 + \varepsilon^{-1}\exp[U_1 - ar^{\nu}\sin(\nu\varphi)\varepsilon^{\nu-1}\delta^{\nu/2}] \tag{1.10}$$
$$a(\tau) = \frac{4^{1-\nu}}{\sqrt{\pi}}\Gamma^{-1}\left(\frac{\nu+1}{2}\right)\tau^{-\nu/2}$$
$$\Delta_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

If the chemical reaction is to have an appreciable effect in the chemical reaction development, i.e. the last term in the right-hand side of Eq.(1.10) is $O(1)$, it is necessary to select

$$\delta(\varepsilon) = \varepsilon^{-2(\nu-1)/\nu} \tag{1.11}$$

Note the considerable dependence of the characteristic time of the chemical reaction process at the wedge tip on the tip angle. From (1.10) we have

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*) V.S. Berman, Certain problems of propagation of the zone of isothermic chemical reactions in gaseous and condensed media. Competition Dissertation for the Science Degree of Candidate of Phys. Math. Sci. Moscow, IMP Akad. Nauk SSSR, 1974.

$$\Delta_r U_1 + \exp[U_1 - ar^\nu \sin\nu\varphi] = 0 \qquad (1.12)$$

The solution $U_1(\tau, r, \varphi)$ must satisfy the boundary condition when $r \to 0$ and remain bounded as $r \to \infty$. This condition is in agreement with the expectation on physical considerations that the most intensive chemical heating occurs in the region surrounding the wedge tip. It is not possible to require the solution of Eq.(1.12) to vanish when $\varphi = 0$ and $\varphi = \alpha$, since the expansion (1.7) and, consequently, (1.10) are valid in a region that excludes any reasonably close approach to the wedge faces.

2. Solution of equation (1.12).

In this equation time is a parameter. We introduce the new function

$$V(\tau, r, \varphi) = U_1 - ar^\nu \sin\nu\varphi \qquad (2.1)$$

Since the second term in the right-hand side of (2.1) is a harmonic function, we have

$$\Delta_r V + \exp(V) = 0 \qquad (2.2)$$

As was first disclosed by Liouville /4/, this equation has a general solution which various forms are given in several publications (see e.g., /5–7/). Here we use a form of solution which differs from the known ones, and which for the solution U_1 of Eq.(1.12) yields the expression

$$U_1 = ar^\nu \sin\nu\varphi + \ln 2 + 2\ln|\nabla_r g| - 2\ln \operatorname{ch} g \qquad (2.3)$$
$$|\nabla_r g|^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \varphi}\right)^2$$

where $g(\tau, r, \varphi)$ is a harmonic function. As shown in /8/, when $\Delta_r g = 0$ then

$$\Delta_r(\ln|\nabla_r g|) = 0 \qquad (2.4)$$

Identity (2.4) proves that expression (2.3) is a solution of Eq.(1.12). It is, thus, necessary to find a harmonic function $g(\tau, r, \varphi)$ such that when $r \to 0$ and $r \to \infty$, the conditions imposed on $U_1(\tau, r, \varphi)$ are satisfied.

Consequently, the solution of the boundary value problem for the nonlinear equation (2.2) (or (2.12)) reduces to finding a harmonic function that satisfies the nonlinear boundary conditions, a problem which can be reduced to solving a nonlinear integral equation.

Let us determine the approximate solution of Eq.(2.2). We seek for g a solution in the form of series

$$g(\tau, r, \varphi) = A(\tau) + B(\tau)\ln r + C(\tau)\varphi + \sum_{i=1}^{\infty} [G_{\pm i}(\tau)\sin(\nu_i\varphi) + F_{\pm i}(\tau)\cos(\nu_i\varphi)]\, r^{\pm\nu_i} \qquad (2.5)$$

where $A(\tau)$, $B(\tau)$, $C(\tau)$, $G_{\pm i}(\tau)$, $F_{\pm i}(\tau)$ are unknown functions that are to be determined using the respective boundary conditions. Taking into account that

$$|g| \to \infty, \quad \ln \operatorname{ch} g = |g| - \ln 2 + o(1)$$
$$r \to \infty, \quad \ln|\nabla g|^2 = \text{const} \cdot \ln r$$

we obtain

$$F_{\pm i}(\tau) = 0, \quad i = 1, 2, \ldots, \quad G_{\pm j} = 0, \quad j = 1, 2, 3, \ldots$$

Consider the particular case of $\nu = 2$ $(\alpha = \pi/2)$. Restricting expression (2.5) to three terms, viz.

$$g(\tau, r, \varphi) = A(\tau) + B(\tau)\ln r + r^2 D(\tau)\sin(2\varphi) \qquad (2.6)$$
$$|\nabla_r g|^2 = B^2(\tau) r^{-2} + 2B(\tau)D(\tau)\sin(2\varphi) + 4D^2(\tau) r^2$$

and setting $r \to \infty$, from (2.6) and (2.3) we obtain

$$U_1(\tau, r \to \infty, \varphi) = [a(\tau) - 2D(\tau)] r^2 \sin(2\varphi) + 2[1 - B(\tau)]\ln r + 3\ln 2 - 2A(\tau) + \ln[4D^2(\tau)] + o(1) \qquad (2.7)$$

To satisfy the condition of boundedness of U_1 it is necessary to set

$$D(\tau) = a(\tau)/2, \quad B(\tau) = 1 \qquad (2.8)$$

Let us now satisfy the condition when $r \to 0$. From (2.3) and (2.6) we have

$$U_1(\tau, r \to 0, \varphi) = 3\ln 2 + 2A(\tau) + o(1) \qquad (2.9)$$

from which

$$A(\tau) = -{}^3/_2 \ln 2 \tag{2.10}$$

$$g(\tau, r, \varphi) = {}^1/_2 [\ln (r^2/8) + ar^2 \sin (2\varphi)] \tag{2.11}$$

From (2.7) we have

$$U_1(\tau, r \to \infty, \varphi) = 2 \ln [8a(\tau)] \tag{2.12}$$

It is obvious on physical considerations that function $U_1(\tau, r, \varphi)$ which defines chemical heating up must be nonnegative. Analysis of solution (2.3) with function $g(\tau, r, \varphi)$ of the form

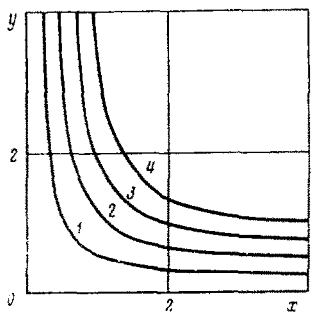

Fig.1

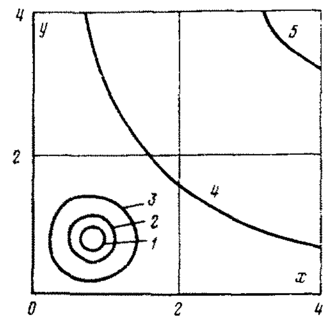

Fig.2

(2.11) shows that this condition is satisfied when

$$a(\tau) \geqslant {}^1/_8 \tag{2.13}$$

Since $a(\tau) = 1/(2\pi\tau)$ when $\nu = 2$, using inequality (2.13) we conclude that solution (2.3) with function $g(\tau, r, \varphi)$ determined by (2.11) is nonnegative when

$$r \to \infty, \; 0 \leqslant \tau \leqslant 4/\pi \tag{2.14}$$

In this case we take as the characteristic ignition time the instant at which the qualitative change of solution

$$\tau_i = 4/\pi \tag{2.15}$$

takes place.

Thus the process of igniting in a region of the form of a straight angle is substantially faster than that of igniting a plane half-space ($\varepsilon \ll 1$). i.e.

$$t_i(\nu = 1)/t_i(\nu = 2) = \varepsilon^{-1}/8 \gg 1 \tag{2.16}$$

Consequently, irregularities of the ignited surface, whose dimensions represent several characteristic zones of steady heating, can play the determining part in the development of the ignition process.

Note that solutions (2.3) and (2.11) do not satisfy the conditions at wedge faces. In these regions it is necessary to introduce new "stretched" variables that are characteristic of the given boundary layer, and construct a solution that satisfies boundary conditions and can be merged with solution (2.3).

An attempt at satisfying both conditions as $r \to \infty$ and $r = 0$ in the case of $\nu \neq 2$ and $\nu = 1$ using formula (2.5) proved unsuccessful. This is explained by the evident necessity to introduce at the wedge tip a complementary "stretched" region, merge it with solution (2.3), and have the condition at $r = 0$ satisfied. The region was not investigated above.

To check the results obtained by asymptotic methods, the input problem (1.5), (1.6) was solved numerically with $\alpha = \pi/2$. Several isotherms are represented in Figs.1 and 2 at various instants of time. The temperature distribution prior to the instant of ignition is shown in Fig.1 with $\varepsilon^{-1} = 40$. Curves 1–4 correspond to $\theta = 0.9; 0.8; 0.7; 0.6$. The behavior of isotherms after the instant of ignition are shown in Fig.2, where curves 1–5 correspond to $\theta = 2.1, 0.7, 0.3, 0.9, 0.5$. The instant of ignition is represented by closed isotherms which surround the hottest region inside the wedge. The time of formation of that "hot point" obtained by numerical

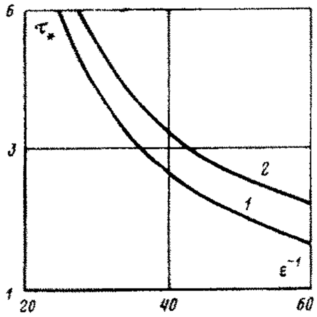

Fig.3

methods for finite values of $\varepsilon^{-1}=O(10)$ is in agreement as to the order of magnitude with the characteristic time (2.15). The dependence of ignition time on ε^{-1} obtained by numerical and asymptotic method is shown in Fig.3 (where $\tau_* = 10^2\tau_i$) respectively, by curves 1 and 2.

The good agreement between the two solutions indicates that the asymptotic method enables us to obtain not only a qualitative picture of the phenomenon but, also, to observe quantitative changes.

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